Asymptotic reasoning and universality in (space)time dynamics

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February 27, 2024

Abstract

The theory of general relativity (GR) displays an array of universality features that are central to the operational use of the theory, such as detecting and extracting information from gravitational waves, while at the same time connecting to foundational issues, e.g. in the gravitational collapse setting or in the high-frequency limit of the theory (in the geometric optics approximation). Quite remarkably, these features can smooth out the details at a certain scale, namely through dissipative/smoothing mechanisms, in a way ensuring some level of predictive power (and simplicity) for the theory even when accurate control of the details is not at reach, either in principle or for operational reasons. We illustrate the role of such universality and dissipative/smoothing mechanisms in a series of examples, with a focus on binary black hole mergers, and we argue for the fruitfulness of (various forms of) asymptotic reasoning for identifying and investigating universal patterns in the GR context. In a broader perspective, we suggest that these latter may provide fundamental insights on the dynamics of (space)time in GR and beyond, in a way that is 'agnostic' to the underlying fundamental (e.g. quantum gravity) ontology.

1 Time and dynamics: an 'asymptotic reasoning' approach

The inquiry about the nature of *time* and, more specifically, its relationship with *dynamics*, understood as the systematic characterisation of the very notion of 'evolution' of a given 'physical system', has driven a substantial part of the development of physics. Our understanding of time has accordingly conformed to the prevailing theory of dynamics.

In this sense, conceptions of time as either 'fundamental' or 'emergent' in physical theories crucially depend on how the notion of evolution is implemented in the respective theories. In this contribution, we show that certain 'operational'—epistemic—features of the spacetime dynamics of the theory of general relativity (GR) can actually provide relevant insights on time, irrespective of whether time is conceived as fundamental or emergent at some deeper level (e.g. quantum gravity or any other alternative fundamental approach to the gravitational phenomenon). To this aim, we will adopt an approach akin to what has been called 'asymptotic reasoning' in the literature [11, 12], where the study of dynamics makes use of asymptotic methods to unveil certain universal patterns (in a precise sense) that may be part of an underlying, more fundamental description.

In section 2, we will make a 'tour d'horizon' of some dynamical features in GR that point towards some sort of dissipation/smoothing mechanisms in the theory (§2.2), starting with the

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challenge of encoding prescribed initial data (§2.1), thus motivating an asymptotic reasoning approach. In section 3, we will consider a specific astrophysical study case, namely binary black hole (BBH) mergers, where asymptotic methods can be applied in various degrees, unveiling a hierarchy of universal patterns. We conclude in section 4, highlighting the fruitfulness of (various forms of) asymptotic reasoning for probing (various types of) universal patterns, possibly hinting at fundamental underlying features. An important lesson of this contribution concerns the epistemic and ontological relevance—in particular for time—of operational considerations about dynamics in the GR setting, where there is no canonical notion of temporal evolution. In the reminder of this section, we briefly set up the conceptual framework of the sort of asymptotic reasoning we shall be using, in particular the related notions of structural stability and universality (relying on [11]).

1.1 The conceptual framework of asymptotic reasoning: structural stability and universality

A basic tenet in our discussion is the focus on universal patterns—universal in the sense of being insensitive to fine-tuned details (see below)—emerging in gravitational dynamics behaviours that play a key role in the predictive power of the theory as well as in the understanding of specific underlying phenomena.

Such a view builds on the notion of "asymptotic reasoning" introduced by Batterman [11], namely an approach to a given theory that explicitly sacrifices precision and exactness by eliminating details, in order to make the underlying patterns apparent. When such underlying patterns display an appropriate form of *structural stability*, they become universal patterns providing the substrate for a notion of *universality* in the theory. Asymptotic reasoning is therefore tightly connected to the notions of 'structural stability' and 'universality'. Structural stability has a precise technical meaning in dynamical system theory, but in a broader perspective it captures the idea that small perturbations (e.g. in the initial configurations or the dynamical equations) do not lead to qualitatively different behaviours of the system under consideration (where 'small' and 'qualitatively different' need to be characterised according to the context). Within the framework of asymptotic reasoning, structural stability may be an indication that physically distinct systems can display similar dynamical patterns, irrespective of their differences—that is: universal patterns. Batterman [11] considers that two general features are characteristic of universality (p. 13):

1. *"The details of the system (those details that would feature in a complete causal-mechanical explanation of the system's behaviour) are largely irrelevant for describing the behavior of interest.*
2. *Many different systems with completely different "micro" details will exhibit the identical behavior."*

In a sense, structural stability (understood in the broad perspective mentioned above) guarantees that these "details" (seen as 'small' perturbations) do not affect the "behavior of interest": it thus provides asymptotic reasoning its epistemic power to identify universal patterns.[1]

Asymptotic reasoning is usually realised in a regime of the theory aiming at focusing on a specific phenomenon and not in the full theory, by filtering away degrees of freedom of the theory not relevant for the phenomenon. From a technical perspective, this is typically characterised by certain limits of a small/large parameter, often entailing a gain in the mathematical tractability and simplicity in the appropriate range determined by the asymptotic parameter.

[1] On the epistemological relevance of structural stability, see the discussion in [30].

Physics and mathematics are rich in examples of asymptotic treatments, ranging from the semi-classical approach to quantum mechanics, the renormalization group, quasi-geostrophic equations in fluids for explaining turbulence and inverse cascades, to the Post-Newtonian series in general relativity and the theory of the rainbow, among many others. The key point is that such a focus on the structurally stable features of specific phenomena in 'filtered' (or 'coarse-grained') regimes do reveal universality aspects of the full theory. As Batterman puts it [11]:

> *"[...] the existence of [structurally stable] solutions––the principal laws or basic features that explain the universal phenomenon of interest––is a reflection of the fact that the process governing the behavior has stabilized in a very important way in the asymptotic regime where the details of the initial and boundary conditions play no significant role."* (p. 59)

In this contribution, we will argue that such stabilization (dissipation/smoothing) mechanisms play a crucial role in the operational articulation of GR as well as in the emergence of universal patterns, and that asymptotic reasoning provides a suitable framework to study these intriguing features of gravitational dynamics.

1.2 Spacetime dynamics as an asymptotic limit: 'operational' gravity in action

The physics of gravitation stands at a crossroads between 'theoretical' and 'operational' problems. On the one hand, as a cornerstone in our understanding of fundamental physics, gravity permeates the conceptual and structural developments of current theoretical physics. The quest of a theory of quantum gravity belongs to this category, but it is not the only setting in which gravity offers a structuring framework for fundamental physics (see e.g. [7]). On the other hand, with the advent of precision cosmology and gravitational wave physics, gravitational physics also plays a crucial 'operational' role both to guide the research and to interpret the observations.

However, there is an explicit tension between both perspectives. Whereas the operational side of gravitational physics explicitly relies on GR and hence seems to fundamentally require spacetime (e.g. curvature) notions, the very need for spacetime structures is under question on the theoretical side of (quantum) gravitational physics.

But even if we stay at the (classical) spacetime level, there are tensions between what we may refer to as a fundamentally geometric perspective and the 'operational' demands of confronting astrophysical systems. The notion of mass in GR constitutes a paradigmatic example: whereas the geometric characterisation of mass involves the full spacetime under specific asymptotic conditions at infinity—and hence cannot be associated with a compact (in space) region of spacetime[2]—the everyday practice in gravitational wave research assigns 'effective masses' (as well as linear and angular momenta) to individual compact objects such as black holes. This is not an academic point, but a very pragmatic issue that is the focus of intense research (e.g. the recent study [66] to provide better local estimates for the angular momentum; see [71, 40] for reviews on the subject).

In this contribution we will focus on a spacetime description of gravitational dynamics, arguing for the fruitfulness of a form of asymptotic reasoning in understanding the emergence of universal patterns that are crucial to the operational side of gravitational physics. Indeed, asymptotic reasoning can be understood as a methodological strategy to address the concrete astrophysical (and cosmological) scenarios that are naturally identified as genuine temporal evolution problems. In this perspective, spacetime (Lorentzian) structures actually provide a sound framework to characterise such evolutions, in particular allowing non-essential degrees of freedom to be 'filtered' in

[2] For a philosophy of science discussion, see [50].

a spirit akin to asymptotic reasoning. The remarkable fact we will discuss in the next section is that, although fundamental obstacles hinder the canonical (geometrical) construction of relevant astrophysical notions—such as (quasi-)local physical quantities mentioned above—a sort of structural stability smooths out the differences, giving rise to universal behaviours that guarantee the operational usefulness of different (and differing) effective estimates.

2 General relativity: the initial value problem and universality

'Dynamical change' can be cast in different schemes in the theory of general relativity. A notion of 'evolution' as the continuous description of the physical system in 'time' is a natural approach when upgrading to the relativistic setting the picture provided by Newtonian celestial mechanics. Given that GR encodes (in geometrical terms) the dynamics of spacetime itself, without an explicit (and canonical) evolution formulation, it is a remarkable feature of the theory that it indeed admits a well-posed evolution description at all. In particular, GR admits a well-posed initial value formulation, that can adopt different realisations, as a hyperbolic (wave) partial differential equation (PDE) system subject to constraints or in the 3+1 Hamiltonian version, each formulation highlighting a structural aspect of the theory. This initial value perspective will play an important role in the following, since it underlies aspects of the class of universal behaviours that we aim at discussing. However, it is important to emphasize that this is not the only option for exploring gravitational dynamics in general relativity. Schemes based on the notion of direct and inverse scattering, built on asymptotic data rather than detailed bulk quantities 'evolving in time', have also played a role in the development of the theory, with a particular renewal in recent times in the setting of gravitational wave physics[3] (cf. e.g. point vii) in section 2.2 and points iv) and v) in section 3.1). Both families of approaches—the 'scattering approaches' focusing on the initial-final states and the dynamical ('on-the-fly') evolution approaches—are intimately connected, but they stress different aspects of the theory. We will now focus on the continuous evolution schemes based on the initial value formulation of GR—although we will see that scattering approaches eventually enter the discussion to shed light on some universality aspects of the dynamics.

2.1 Initial value problem in General Relativity

The initial value formulation is a very natural approach to study the dynamics of point particles in Newtonian dynamics, both from the perspective of the physical characterisation of the system and for the mathematical analysis of the dynamics. The latter involves both the aspect of describing the evolution of a given system, as well as the question of analysing its stability when varying initial conditions (namely when studying the impact of performing variations in the phase space). The same remains true when upgrading the theory from particles to fields, but still with a privileged global time providing a canonical notion of simultaneity and an instantaneous 'synchronisation' mechanism through some 'action at a distance'. In all these (non-relativistic) contexts, setting initial conditions at a given time is a natural and unproblematic procedure.

The situation is less clear in the context of a relativistic theory, in particular in theories of spacetime dynamics: whereas the mathematical question is well-posed and leads to important insights into the structure of the theory, from a physical perspective it is not given for granted that the initial value approach provides the natural scheme to study the dynamics of spacetime from a fundamental perspective (although it may prove very useful for a number of purposes).

[3] See e.g the cross-correlation approach to strong-field spacetime dynamics proposed in [47, 48, 38, 46] and further developed in works such as [34, 63, 37], the 'gravitational wave tomography' program in [10, 9] or some of the elements in the celestial holography framework, cf. [70]).

The mathematical consistency of the initial value problem for 'well-posed' theories of spacetime dynamics is guaranteed in the following sense: prescribing appropriate initial data on an initial manifold Σ_0 fixes completely, up to the so-called *Cauchy horizon*, a spacetime (M, g_{ab}) where the manifold M is built up in terms of the slicing $\bigcup_t \{\Sigma_t\}$, where slices Σ_t are uniquely evolved from data at the initial slice Σ_0.[4] The well-posedness of such initial value problem for GR has been proven by the foundational results of Choquet-Bruhat and Geroch (cf. [31, 21]). In this context, one often refers to globally hyperbolic spacetimes, which are spacetimes (M, g_{ab}) admitting a so-called *Cauchy hypersurface* Σ_0 from which the spacetime can be reconstructed as a maximal future development of an initial data set.[5]

Data at the initial slice are provided by the pair (γ_{ij}, K_{ij}), where γ_{ij} is a Riemaniann metric in Σ_0 and K_{ij} is a symmetric tensor to be eventually interpreted as the extrinsic curvature of (Σ_0, γ_{ij}) into (M, g_{ab}). The pair (γ_{ij}, K_{ij}) must satisfy (in vacuum) the so-called constraint equations

$$R + K^2 - K_{ij}K^{ij} = 0 \tag{1}$$

$$\nabla_j(K^{ij} - \gamma^{ij}K) = 0 \ , \tag{2}$$

where $K = \text{Tr}_\gamma K_{ij}$, ∇ is the Levi-Civita covariant derivative associated with γ_{ij}, with Ricci scalar R. The first equation is referred to as the Hamiltonian constraint, whereas the second is the momentum constraint. Once these data are prescribed, different schemes exist in order to provide a detailed account of the evolution building a unique (maximal future development) spacetime, as well as a methodology to analyse the impact of changing the initial data (i.e. to probe the phase space of the system).[6] Therefore, in this mathematical sense, the situation is similar (and this is a remarkable non-trivial fact) to what we commented above for the initial data problem in Newtonian particle dynamics. A particular qualitative different feature in the general relativistic setting is that the 'time function' t slicing the spacetime, is constructed along the evolution process in connection with the Hamiltonian constraint (1), and is not unique.[7]

However, when considering the problem from a physical perspective, the situation departs from that of Newtonian physics in at least two important ways: first, in the very sense given to the global time t, and second, in the ability to prescribe the physical content 'ab initio' at a given initial time.

Regarding the first aspect, in the spirit of the initial data strategy one should be able to set initial data in a Cauchy slice Σ_0 providing a particular notion of 'simultaneity' at an initial time $t = t_0$. However, this is operationally a challenge in a relativistic theory, and the very schemes one could devise to 'synchronise' data at a given (initial) time suggest that this is not the natural approach. There is simply no unique and natural notion of simultaneity, and there is no (operational) way of setting data 'instantaneously' at spatially separated distances. A smooth 'temporal function' t, as in [13], provides an ordering structure for data that are correlated (constrained) at a given constant time t, but it does not provide an actual physical magnitude that we could call 'time' in

[4]Here and in what follows, 'completely' and 'unique(ly)' should be read 'up to diffeomorophisms'—which does not affect our argument.

[5]Note that not all spacetimes admit such a Cauchy hypersurface, and therefore not all spacetimes are a priori accessible as evolution of appropriate initial data. In this setting, the notion of strong cosmic censorship (e.g. [8]) ensures that spacetimes cannot be extended beyond their *Cauchy horizon*, namely the boundary of the maximal future development. Globally hyperbolic spacetimes are thus the only ones relevant from an evolution point of view.

[6]A particularly enlightening description of this aspect is the account in [29], where the ADM version of Einstein equations are set as a dynamical system in a phase space, and the dynamics is explicit written in terms of the coadjoint of the differential of the constraint functions. This underlines the crucial role of equations (1)-(2), not only to constraint initial data, but also to determine the dynamics.

[7]It is also a highly non-trivial feature of globally hyperbolic spacetimes that they do admit a 'temporal function' t that is not only continuous but differentiable [13]. If such a function does not provide a unique, global time, it does however play a key structuring role in the theory.

the sense that can be attributed to observers using clocks (although in a stably causal spacetime the function t is strictly increasing on any future–directed causal curve [35]).

Regarding the second aspect, a conceptual and technical challenge actually arises when prescribing the pair (γ_{ij}, K_{ij}), aiming at encoding the physical content of the system one wants to study. The difficulties of this prescription step are ultimately rooted in a form of 'non-locality'[8] and the non-linearity of the PDE systems obtained from the geometric constraints (see the detailed discussion in [44]). In particular, prescribing individual masses, angular momenta and other physical quantities of various compact objects interacting dynamically is a question not admitting a single canonical answer [71, 40]. Different prescriptions do exist (except for global quantities, typically at infinity for isolated compact systems) and all of them provide a consistent picture, as long as we keep track of their correspondences when comparing initial data. In addition, because of the nature of the problem to be solved for constructing consistent initial data, it is very difficult not to include 'junk' initial data that is unrelated to the system under consideration.[9]

But the theory displays a remarkable feature: the resulting dynamical system is quite insensitive to the variations among different prescriptions for encoding some physical content (e.g. corresponding to some observational data) and, more importantly, it efficiently eliminates the features associated with the junk part of the initial data (cf. point viii) in section 2.2). A dynamical spacetime 'dissipation/smoothing mechanism' occurs, coarse-graining the dynamics and dissipating the tiny differences between alternative effective descriptions of the same physical system. Such a 'smoothing out' phenomenon can thus be understood as an instance of structural stability, and, given the difficulty to encode sharp prescribed features in the initial data, it turns out to be crucial for the predictive power of the theory [44].

In summary, we retain two important points from the discussion above about the initial data ('evolution') perspective on spacetime dynamics:

i) Spacetime dynamics incorporates a dissipation/smoothing mechanism that efficiently coarse-grains over (a sector of) the dynamical degrees of freedom, smoothing out their evolution.

ii) A (smooth, non-unique) time function t can be constructed permitting an initial value formulation of spacetime dynamics in which a spacetime dissipation/smoothing mechanism guarantees the structurally stability of the initial value scheme by smoothing out non-essential 'sharp' features (e.g. the junk part of the initial data). The function t does not correspond to any actual observable to be measured by physical clocks,[10] but rather constitutes a tool to parametrise 'continuous change' (evolution): for instance, it enables the encoding of correlations between different points in spacetime in terms of the system of constraint equations.

2.2 Spacetime dissipation/smoothing mechanism

Although the effectiveness of underlying dissipation/smoothing mechanism is crucial in the common application of general relativity to relativistic astrophysics, its precise nature is difficult to pinpoint and its remains rather elusive. It is at this point that asymptotic reasoning proves fruitful, by

[8]In the standard approaches to solve the constraints, this 'non-locality' stems from the elliptic nature of the problem on Σ_0; other (e.g. parabolic) approaches can however be envisaged, see [44] for a discussion.

[9]Following [44], we stress that these difficulties in constructing prescribed initial data have an epistemic rather than ontological nature: along the evolution of any system building up a dynamical spacetime, the constraint equations (1)-(2) are identically satisfied at each (arbitrary) spatial slice Σ, since geometrically they correspond to the Gauss-Codazzi equations guaranteeing that slices build up a spacetime. The difficulties mentioned here are related to the epistemic need to construct initial data according to a given prescription. See however point viii) in section 2.2.

[10]It is a growing function along the trajectory of any future observer though [35].

considering different asymptotic regimes and limits of the theory and unveiling the underlying patterns.

We list below some instances illustrating (various forms of) this dissipation/smoothing mechanism, providing simpler dynamics than the one expected a priori (focusing on compact objects in general relativity)[11]:

i) *Post-Newtonian 'unreasonable effectiveness'*: in the description of the two-body problem in general relativity, the Post-Newtonian formulation provides an asymptotic expansion describing the regime in which both velocities and self-gravity of compact objects are both commensurate and small, namely aiming at modelling the 'inspiral phase' in compact binaries. The surprising and unexpected fact is that the scheme continues to provide an excellent approximation beyond its limits of validity [76], well into the fully non-linear 'merger phase'. The binary dynamics is simply smoothed-out at the merger phase.

ii) *Effacement property or 'Effacing Principle'* [27]: still in the context of the inspiral phase of binaries, a remarkable simplification occurs in the dynamics, namely the fact that the internal structure of the compact object starts to be accessible only at at high order in the expansion and it is effectively hidden till late stages when tidal effects start playing a role. This translates in the fact that compact objects can be just parametrised by mass M and angular momentum J. This effacement of the internal structure is associated with the (strong) equivalence principle in general relativity [77].

iii) *Effective-one body approach to binaries*: in addition to these features, further elements in the Post-Newtonian description converge to build a remarkable simple picture in which the full relativistic dynamics is captured in terms of the dynamics of a single body under an effective potential, namely passing from an in principle complicated non-linear PDE to a much simpler ordinary differential equation picture [28].

iv) *Close-limit approximation and effective linearity*: the so-called 'close-limit' approximation of BBH mergers introduced by Pullin and Price [65] shows that the merger phase is very well captured by the perturbation of a single black hole. On the other hand, perturbation theory provides an appropriate mathematical setting to describe the late time ringdown phase. In this sense, the 'close-limit' approximations supports the validity of the ringdown perturbative treatment beyond its a priori range of validity. This is complementary, from the post-merger side, to the to the 'unreasonable effectiveness' of the Post-Newtonian scheme in the inspiral pre-merger phase. In sum, the whole binary dynamics including the 'non-linear' merger phase is 'smoothed-out' giving rise to a much simpler picture than expected with an 'effective linearity' (cf. discussion in [39]).

v) *Gravitational collapse, cosmic censorship and 'no hair' theorem*: the picture of gravitational collapse in classical general relativity is constituted by a series of theorems and conjectures providing an interesting example of asymptotic reasoning. Starting from Penrose singularity theorem [59] guaranteeing the formation of a singularity under very generic conditions (and crucially oblivious to details), the cosmic censorship conjecture [60, 61] then proposes the formation of a horizon causally disconnecting the singularity from a distant observer. Under a second (stability) conjecture, general relativity dynamics drives this isolated system to

[11]In a cosmological setting, one could wonder if the homogeneity at large scale could respond also to a dissipation/simplication mechanism. This will be addressed elsewhere.

stationarity.[12] Finally, under the assumptions of stationary, vacuum and a (Killing) horizon, the black hole uniqueness theorem (see e.g. [22]) establishes that the resulting compact object is a (sub-extremal) member of the Kerr spacetime family. This implements a black hole 'no hair' result where the final state only depends on two parameters, namely the mass M and angular momentum J of the black hole. This resonates with Damour's effacing property, in a picture where dissipation (at infinity and at the horizon) as well as the formation of a positive mass (stable) self-gravitating compact object play key roles. The whole framework instantiates a dramatic example of asymptotic reasoning where a universal pattern emerges (in the late time asymptotic limit), independently of any fine details of the initial data or the dynamics.[13]

vi) *Redshift effect and stability*: a crucial element in the dynamical picture above is the 'red-shift effect' [25, 26] that provides a stabilizing mechanism for the wave dynamics in black hole backgrounds. In particular, the role of the black hole horizon as a dissipation element becomes explicit through the 'local redshift effect'. This latter provides the key geometric property that controls the strongly dissipative character of black hole quasi-normal modes, by structuring the quasi-normal frequencies in the complex plane into bands of width given by the surface gravity κ [75]. This guarantees the fast decaying time-scales of black hole overtones, as well as controlling the universal Weyl asympotics of black hole quasi-normal modes [45].

vii) *Spacetime dynamics 'cross-correlation' effectiveness*: the approach to spacetime dynamics [47, 48, 38, 46] based on the correlations between geometric quantities at an outer (e.g. null infinity) and inner (e.g. black hole horizon) 'screens' also strongly relies on an effectively (approximately) linear dynamics of the propagating gravitational fields in the spacetime bulk. Why such a linearity assumption holds in the appropriate regime is not clarified in this scheme.

viii) *Numerical relativity initial data and 'junk radiation'*: constructing initial data numerically in a initial value problem for spacetime dynamics involves the numerical resolution of constraint equations. Loss of accuracy in such a resolution 'contaminates' initial data generating what is known as 'junk radiation'. However, such junk radiation is typically very effectively radiated away leaving a consistent filtered constraint-preserving system.[14] This represents a practical instance of asymptotic reasoning in action, relying on a dynamical mechanism that again exploits an effective linearity.[15]

[12]The positive mass theorem suggests that, under 'physically reasonable' dynamics, an isolated system, in which no energy is injected and dissipation occurs at infinity and at the horizon, should reach a stationary regime. But what those specific physically reasonable dynamics is not clarified at this stage, constituting one of the main open issues in mathematical relativity, namely the complete proof of black hole stability (cf. [25, 26] for the role of the 'red-shift' effect as a stabilizing mechanism).

[13]The role of 'universality' and 'robustness' when semi-classical effects are incorporated in this black hole setting, giving rise to the so-called Hawking radiation [36], has been systematically addressed in [33]. The authors conclude that the involved 'trans-Planckian' problem, that threatens the inner consistency of the semiclassical treatment in this black hole setting, is not appropriately addressed by any of the proposed universality arguments, in stark contrast with the successful Wilsonian arguments in the context of condensed matter. This criticism does not affect our classical discussion, that can be seen as an instance of 'soliton resolution' [15] in the gravitational setting (see point iv) in section 3.1).

[14]For a review of the issues at the interface between numerical and mathematical general relativity, see e.g. [62, 49].

[15]We note that, as pointed out by their promoters, the practical application of the 'gravitational wave tomography' program [10, 9] to numerically constructed spacetimes relies, at least to a certain extent, on such efficiency to eliminate junk radiation.

ix) *Geometric optics approximation and high-frequency limit in general relativity*:[16] general relativity is indeed a very special theory, not just any spacetime dynamics theory. Although it is a non-linear PDE theory, non-linear terms enter the picture in a very special way, which, in combination with the (diffeomorphism) gauge invariance, makes the theory effectively more linear than a priori expected. This linearisation phenomenon is explicitly seen in the geometric optics approximation of general relativity, namely the asymptotic limit of high-frequencies. Specifically, wave dynamics in the geometric optics scheme gives rise to a set of transport equations for wave profiles that, in the weakly non-linear regime, generically include non-linear terms. However, in the case of Einstein equations, a special phenomenon[17] occurs and non-linear terms finely compensate, rendering the transport equation linear even in the (weakly) non-linear regime. This remarkable feature of general relativity (in particular preventing waves in spacetime dynamics from breaking as sea waves do on the coast) was identified by Choquet-Bruhat [19, 20], recently further systematized and extended in [72, 73].[18] In the very spirit of the asymptotic reasoning pursued here, this suggests the possibility that this effective linear universal pattern identified in a (geometric optics) asymptotic limit corresponds to a structural underlying feature of the full theory. This would provide a non-trivial insight into the various instances of 'effective linearity' presented above.

x) *Universality and simplicity of the BBH merger waveform*: in [43, 41, 42] a scheme based on asymptotic reasoning is proposed to address the unexpectedly simple (and universal) structure of binary merger gravitational waveforms. This will be sketched in section 3.

The previous points suggest the presence of an underlying mechanism in GR that smooths out small details, providing a substrate for structural stability and hence for universal behaviours in a GR regime relevant to astrophysics (and cosmology). This mechanism seems to provide a commensurate balance between efficient dissipation[19] and the realisation of the (strong) equivalence principle in self-gravitating objects, mediated by an enhanced effective linearity in wave propagation.

3 Binary black hole mergers and universality: a study case

In the previous section we have presented general relativity as an initial value problem and have argued that a type of structural stability, involving a dissipation/smoothing mechanism, plays a crucial role in building the concrete predictive power of the theory. Such a perspective relies on an asymptotic reasoning scheme, focusing on specific phenomena through the elimination of non-essential details. In this sense, and in order to complement the instances of the dissipation/smoothing mechanism presented in section 2.2, we focus in this section on a specific phenomenon, namely the point x) above concerning the binary merger waveform.

[16]We are fully indebted to Arthur Touati for his patient and detailed explanations on this point. We owe our insights here to him, although misinterpretations of the mathematical results are our sole responsibility.

[17]Specifically, denoting formally by g the spacetime metric, semi-linear terms $\sim (\partial g)^2$ can be absorbed into quasi-linear terms $\sim g\partial^2 g$, that can then be eliminated by an appropriate choice of (generalised harmonic) gauge. This is a very special feature of general relativity. Interestingly, this resonates with the discussion in [32] about an effective virialization phenomenon in the appropriate regime of general relativity.

[18]The here discussed phenomenon is known as *transparency* in the geometric optics setting. [72] provides a sounder basis to the geometric optics treatment of general relativity in [19, 20], in particular applying it to prove a version of Burnett's conjecture [16] on spacetime high-frequency limit.

[19]In particular, dissipation occurs at the spacetime 'boundaries', including at horizons when black holes are present. This means that the dissipation/smoothing mechanism that we are discussing is in principle restricted to the causally accessible outer domain, therefore not including black hole interiors. Non smooth GR phenomena such as spikes in the setting of the BKL conjecture are therefore not affected by this mechanism. Cosmic censorship seems to play a crucial role for this particular discussion of asymptotic reasoning in the gravitational context.

The dynamics of compact object binaries and its associated gravitational wave emission, displays three phases: 1) an inspiral phase where the two compact objects orbit each other and slowly approach as they lose energy by gravitational wave emission, 2) a merger phase, dominated by the non-linear aspects of the theory, where the two compact objects coalesce into a single compact object and, 3) a relaxation phase where the remnant rings down and approaches stationarity in a regime controlled by linear perturbation theory. Early educated expectations (cf. e.g. Kip Thorne's cartoon in fig. 1 of [68]) suggested an initial sinusoidal signal with slow amplitude increase in the inspiral phase, followed by a violent merger phase (a 'spacetime storm') and ending at the late ringdown in an exponentially damped sinusoidal signal determined by its characteristic (complex) quasi-normal frequencies. Although the inspiral and the ringdown where finely captured by the Post-Newtonian and perturbative treatments, respectively, the merger phase expectations completely failed to match the obtained numerical results first [64], and the gravitational wave observations later [1]: in stark contrast with the expected violent features, the merger waveform presents a smooth and simple interpolation between the inspiral and the ringdown. And the signal is not only simple, but also universal in the sense of being largely independent of initial data (see the discussion on universality in section 1). At the sight of this unexpectedly simple behaviour, a natural question raises:

Is the binary black hole merger waveform boring or elegant?

Attitudes towards this question typically differ according to the background and interests of the researcher. Whereas astrophysicists tend to see such a signal as 'boring', especially when comparing it with the 'rich' signal associated with neutron stars mergers (much more complex due to the matter equation of state), theoretical physicists tend to see it as 'elegant', its simplicity suggesting an underlying mechanism in the theory that prevents complex behaviour from occurring. Here we adopt this latter approach, namely, the view that the simplicity and universality of the signal provide a probe into the fundamental structures of (a sector of) the theory.

3.1 Asymptotic reasoning in the BBH merger waveform: from universality to integrability

In [43] (see also ([41, 42]) a bottom-up approach is proposed to explore the simplicity and universality of the signal in an attempt to identify the underlying structurally stable mechanisms. This effort is structured in a hierarchy of layers, each one providing a (horizontal) instance of asymptotic reasoning and the full hierarchy providing itself a (vertical) instance of asymptotic reasoning, since each new layer adds some details filtered in the previous one. The asymptotic reasoning hierarchy is displayed in table 1, where for each model the mathematical/physical framework is presented, as well as the underlying key structures and mechanisms.

Asymptotic BBH Model	Mathematical/Physical Framework	Key Structures/Mechanisms
Fold-caustic model	Geometric Optics Catastrophe (singularity) Theory	Arnol'd-Thom's Theorem Classification of Stable Caustics
Airy function model	Fresnel's Diffraction, Semiclassical Theory	Universal Diffraction Patterns in Caustics
Painlevé-II model	Painlevé Transcendents and Integrability Self-force calculations and EMRBs	Painlevé property Non-linear Turning Points
KdV-like model	Inverse Scattering Transform and Integrability Dispersive Non-linear PDEs Critical Phenomena in Dispersive PDEs	Painlevé test, Lax pairs Darboux transformations, Soliton Scattering Universal Wave Patterns, Dubrovin's Conjecture
Propagation models on (anti)-Self-Dual backgrounds	Ward's Conjecture and Integrability Twistorial techniques	(anti-)Self-Dual DoF, Instantons, Tunneling Penrose Transform, 'Twistor' BBH data

.

Table 1: Asymptotic reasoning hierarchy of models for the BBH merger dynamics (from [43]).

In the following we sketch the ideas underlying this hierarchy (and refer to [43, 41, 42] for more details):

i) *Gravitational wave detection as a caustic in time*: this presents a fully minimalistic model for detection, by considering a geometric optics scheme in which gravitational radiation is modelled by rays. In such a framework, caustics, that can be understood as the enveloping of rays, are generic and, once formed, structurally stable (Arnol'd-Thom theorem). The important point is that 'in the interior of the caustic' the intensity of the signal increases as the caustic is approached, diverging at the caustic and then dramatically falling down (or disappearing in the case of the so-called fold caustic). This minimalistic model proposes that a caustic phenomenon, occurring in time and controlled by the fold generating function (phase) $\phi_{\text{fold}}(t,s) = ts + \frac{1}{3}s^3$, with s a 'state variable' characterising the system, underlies the signal detected by the interferometer located at some 'fixed space point': the detector receives a signal whose intensity growth with a Post-Newtonian pattern correctly captured by the fold caustic $\phi_{\text{fold}}(t,s)$ properties, diverging at the coalescence and fully vanishing afterwards, so that there is no hope to include the ringdown. The merits of this basic model is to clean up almost all details except a minimalistic description in terms of rays, to be universal thanks to caustic structural stability and to provide the skeleton for the next diffraction model.

ii) *Caustic diffraction patterns and Airy function BBH waveform model*: the second layer adds a new physical ingredient beyond rays, namely it incorporates the wave-nature of the radiation but without resorting the full theory, but only keeping the minimal wave elements in the form of diffraction theory. In a slogan form: if caustics provide the skeleton, diffraction provides the flesh [14]. On the one hand this regularises the caustic diffraction and, on the other hand, it provides a universal and structurally stable (caustic) diffraction pattern. The fold caustic leads to the so-called Airy function

$$h(t) \sim \text{Ai}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} ds\; e^{i\phi_{\text{fold}}(t,s)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} ds\; e^{i(ts+s^3/3)}). \tag{3}$$

This extremely simple Airy model [41] for the gravitational signal $h(t)$ detected at the gravitational antenna captures remarkably well the binary-merger phases. From the asymptotic reasoning perspective, the universal pattern that is unveiled in this high-frequency asymptotic limit is the Airy function. More specifically, the underlying Airy equation, namely $\ddot{h} - th = 0$, is the actual universal pattern that is identified, with its structural role as a 'turning point' equation interpolating between an oscillatory (inspiral) regime and an (over-)exponentially damped one after the merger, together with its built-in dispersive character.

iii) *Painlevé-II turning point model and EMRIs*: the Airy function does not allow to account for the transition to the ringdown phase, a non-linear term being needed to account for the behaviour of the frequency at the coalescence time. Pushing along the turning-point aspect, the archetype of non-linear turning point generalisation of the Airy equation is the so-called Painlevé-II equation [3], namely $\ddot{u} - tu - 2u^3 - \alpha = 0$. A remarkable feature is that such an equation plays indeed an explicit role in the exact solution of the self-force problem of a charged particle in a Coulomb field [67], suggesting its relevance for the BBH problem in the asymptotic limit provided by extreme mass ratio inspirals (EMRI) [42]. The universal pattern extracted from this asymptotic limit would be the Painlevé-II equation. But a crucial remark needs to be made here: Painlevé-II and more generally the six Painlevé transcendents [23, 24] are a smoking gun of an *integrable* structure. The universal pattern identified at this level is the Painlevé-II structure and its associated integrability, which threads the BBH dynamics from the inspiral to the rigndown, including the intermediate transient merger, and which grounds its universality and simplicity.

iv) *Non-linear dispersive PDE integrability, universal wave forms and 'soliton resolution'*: in the next asymptotic level, new elements are incorporated in the model to upgrade Airy/Painlevé-II ordinary differential dynamics to a PDEs description. A crucial methodological step is enforced at this stage, aiming at reconciling the success of the effective linear dynamics referred to above (see the points iv), vii), viii) and especially ix) in section 2.2) with the non-linear character inherent to Painlevé-II integrable systems and the associated universal wave forms appearing in such non-linear dispersive settings [58]. We adopt a 'wave-mean flow' perspective on dynamics by separating fast and slow degrees of freedom:

  a) Fast degrees of freedom with effective linear dynamics, propagating on a slowly dynamical (non-linear) background and reaching infinity where the BBH waveform is observed.
  b) Slow degrees of freedom subject to a non-linear (quasi-)integrable dynamics instantiating the Painlevé-II structure and providing the background for the fast degrees of freedom.

  Integrability is therefore not upgraded to the full general relativity theory, but only to a (slow) sector of it. The claim that integrability only affects a sector of the theory is crucial and has important consequences. On the one hand, it is not clear that in this 'wave-mean flow' scheme the dynamics should be fully treated in the 'initial value' formulation detailed in section 2; rather, an appropriate combination of initial value and 'direct/inverse scattering' approaches may be more appropriate (see e.g. [10, 9, 70]). Scattering approaches often take the best out of integrability, in particular by identifying hidden symmetries in the system, such as the Darboux symmetries in [53, 54, 56, 55, 57]. On the other hand, the '(linear) fast/(integrable) slow' split offers an asymptotic methodological avenue to probe a dynamical feature that probably underlies the full theory and is known as the 'soliton resolution' conjecture [15]: generic global-in-time non-linear wave dynamics tends to decouple universally at late times into 'solitons' plus 'radiation'. This behaviour constitutes an archetype of asymptotic reasoning, since the initial details are completely erased in the process, only remaining the final radiation scattering over 'solitons', special attractor solutions in the theory phase space. In this sense, the gravitational collapse picture in point v) of section 2 provides an archetype of soliton resolution in the general relativity gravitational context.

v) *Scattering in non-linear (integrable) self-dual backgrounds*: the previous layers have built a bottom-up asymptotic reasoning hierarchy for BBH dynamics by progressively adding new details, leading to a picture of BBH dynamics in which fast degrees propagate according to linear wave dynamics over a non-linear (dispersive) integrable PDE system. At this point, it is natural to wonder if this fast/slow 'wave-(integrable) mean flow' picture can be reached from the full Einstein equations via a top-down approach in which details are actually filtered out, rather than added. It is in this setting that the so-called Ward conjecture [74, 2] enters the picture, by providing a link between integrable systems and self-dual solutions in Yang-Mills and GR theories. More specifically, in the spirit of asymptotic reasoning, one would aim at starting from Einstein equations in the BBH setting and filter non-essential elements in order to cast the dynamics in terms of the scattering of fast degrees of freedom over bound-states of the self-dual sector of the theory. Two remarks are relevant here: a) self-dual integrable solutions in GR are either complex or are real solutions in the Euclidean counterpart of (Lorentzian) GR, that is, we are naturally led to address BBHs as a scattering problem over gravitational instantons; and b) twistor theory is tailored to deal with self-dual structures in

GR, making twistors the natural concept to characterise the phase space of integrable slow degrees of freedom in the BBH problem.

3.2 Fundamental gravitational structures from asymptotic reasoning

In the previous subsection we have presented a 'study case' to illustrate how, starting from a particular phenomenon (in our case the simplicity of the BBH merger waveform), one can use an asymptotic reasoning to probe the fundamental structures of the underlying theory, by unveiling universal patterns in an asymptotic limit. The claim is that such universal patterns reveal, beyond that particular asymptotic limit, a structure that may actually underlie the full theory well beyond the asymptotic limit where it becomes apparent. Specifically, the previous BBH case suggest the following structural elements underlying the full theory:

i) *Integrability of a sector of the theory: Painlevé-II structure.* The universality of the BBH merger waveform would be a manifestation of an underlying integrable sector of GR. Integrability is not claimed for the whole GR, which is known not to be integrable, but for the sector of the GR phase space associated with (solitonic) 'background' degrees of freedom. From a technical point of view, Painlevé-II transcendents would play a key role in this kind of dynamical transients.

ii) *Hidden symmetries in the bulk spacetime.* Intimately associated with this integrability controlling soliton backgrounds, a set of hidden symmetries can underlie the dominating structural features of the dynamics in the bulk (in the spirit of Darboux' transformations identified by Lenzi & Sopuerta [53, 54, 56, 55, 57] and associated with infinite dimensional Lie algebras in the Korteweg-De Vries hierarchy). How these bulk symmetries interplay with asymptotic symmetries is a topic of current research pioneered by Lenzi & Sopuerta.

iii) *Twistor and self-dual structures.* A most surprising outcome of the asymptotic reasoning discussion of the BBH merger waveform universality is the proposal that "twistor data" could parametrize the self-dual sector of the theory controlling the non-radiative degrees of freedom of the theory, with a version of the Penrose transform potentially providing an integral representation of such slow background solutions. This is complementary to recent approaches to study perturbations on self-dual backgrounds [4, 6, 5], although our scheme is a more humble one not attempting to provide a comprehensive formalism, but just aiming at identifying the fundamental nature of the relevant degrees of freedom in the problem.

iv) *Time and times.* As we have argued at the beginning of section 1, the notion of time critically depends on the physical theory that is considered (as well as the way in which time is encoded in this theory). In our discussion, time has entered the initial value problem of GR as a 'temporal function' that permits to label spatial slices, providing a methodological tool to parametrise the transition between 'initial' and 'final' states of the system. It is unclear that such a function t represents a quantity that can be associated with what the 'clock' of an observer would measure, and it rather identifies sets of points in spacetime that are subject to (Einstein) constraints. But such a parametrised description has been key to unveil underlying structures of the theory, in particular through a methodological separation into fast/slow degrees of freedom, suggesting that the notions of time/clock are not necessarily unique, even in the same dynamical framework. Furthermore, the slow degrees of freedom might be better described by a scattering picture, where the continuous time description ('on-the-fly') is not even needed. The bottom line is that, in our initial value discussion, time appears as an epistemic rather than an ontological notion.

4 Beyond spacetime dynamics: insights from universal patterns

The various examples discussed in section 2.2 as well as the BBH merger study case in section 3 show the fruitfulness of (the various forms of) asymptotic reasoning in the general relativistic context, in particular when it comes to unveiling certain universal patterns. Of course, each example of section 2.2 would require a detailed treatment—in this sense, our work here is very much programmatic. Identifying and understanding these universal patterns as well as the underlying (dissipation/smoothing) mechanisms connect to many important foundational topics in the theory of general relativity (as the examples of section 2.2 testify)—and possibly beyond. Indeed, to the extent that these universal patterns are 'insensitive' to the underlying 'details' of a possible more fundamental (e.g. quantum gravity) theory, they may provide fundamental insights into dynamics and time in the gravitational context—whether or not (space)time turns out to be fundamental or emergent in some (e.g. functional [51] [52]) sense, according to a more fundamental description). The dissipation/smoothing mechanisms that the different GR features discussed in this contribution seem to point at constitute a striking example.

As we mentioned in section 1.1, the conceptual framework of asymptotic reasoning, whose fruitfulness in GR we are arguing for, is directly inspired by the work of Batterman [11]. However, we are not committed to his argument that universal features constitute evidence for a form of emergence and against (e.g. Nagelian) reduction. In fact, following Butterfield [17] [18] (and others), universality and emergence need not stand in opposition to (Nagelian) reduction: the idea is that emergence can be understood in terms of novel and robust behaviour (relative to some comparison class), which need not be incompatible with reduction. Similarly, we consider neither the (forms of) asymptotic reasoning nor the universal patterns discussed here in the GR context to stand in tension with reductionism. On the contrary, our main motivation is to probe fundamental structures in the theory by adopting an asymptotic reasoning methodology (as explicitly illustrated in the integrability proposal from the observed universality in BBH waveform mergers [43, 41, 42]). This can be realised in complementary bottom-up (first sequence) and/or top-down (second sequence) approaches:

$$\text{Asymptotic reasoning} \rightarrow \text{Universal patterns} \rightarrow \text{Fundamental structures},$$
$$\text{Universal patterns} \leftarrow \text{Asymptotic reasoning} \leftarrow \text{Fundamental structures}.$$

But more work remains to be done. Indeed, it is important to stress that the various examples discussed in sections 2.2 and 3 refer to various forms of asymptotic reasoning as well as to various types of universal patterns, possibly underlaid by a variety of what we have called 'dissipation/smoothing mechanisms'. For instance, compare the forms of asymptotic reasoning at work in gravitational collapse (point v) in section 2.2) with the geometric optics approximation (point ix) in section2.2), or compare the bottom-up with the top-down approach in the asymptotic reasoning hierarchy for the BBH merger waveform (section 3.1). Both the variety and the common features call for further investigations.

We would like to end with two interrelated comments. First, part of the foundational considerations in this contribution arise from the operational side of GR, as the BBH merger study case testifies: casting the theory in terms of an initial value problem, running numerical relativity experiments and making sense of (the simplicity and universality of) the observed BBH merger waveform highlight the relevance of (various forms of) asymptotic reasoning and of underlying dissipation/smoothing mechanisms, which, as we have argued, deeply connects to foundational features of GR and beyond. More broadly, this highlights the relevance of epistemic considerations—sometimes of quite a pragmatic nature—at the non-fundamental level of GR for foundational (and ontological) issues about (space)time. One important lesson here is that there is still much to be learned about spacetime at the level of classical GR and its operational articulation—when it comes to understanding the nature of spacetime, it is not all about quantum gravity.

This leads us to our second point: asymptotic reasoning in the GR context does not merely amount to a set of nice mathematical methods—e.g. allowing to enhance mathematical tractability and physical operationalisation (very important tasks in themselves!)—but also fundamentally involves what has been called an "exploratory role" in the literature (see section 4 in [69] as well as references therein). As Shech puts it (writing about "infinite idealizations", to which asymptotic reasoning belongs) [69]: "it is suggested that infinite idealizations may play an indispensible role in the exploration of scientific theories, intertheoretic relations, and phenomena, and that possibly such exploration is needed for deep sense of scientific understanding" and that, in addition, " infinite idealizations do seem to play an indispensible role in exploring the possible representational structure and foundations of scientific theory" (p. 11)—here the theory of general relativity.

5 Acknowledgments

We would like to thank Piotr Bizoń, Carlos Barceló, Badri Krishnan, Michele Lenzi, Mikhail Semenov-Tian-Shansky, Carlos F. Sopuerta and Arthur Touati for illuminating discussions. We thank Marcus Ansorg for his praise for elegance, consistency and beauty of general relativity, and his lessons about "General Relativity's teeth". We also thank the participants to the conference "Perspectives on Time and Atemporality in Cosmology and Natural Sciences" at the University of Milan in July 2023 for the interesting discussions, and in particular Silvia De Bianchi and her team for their work and patience. This paper is dedicated to the memory of Suzanne Lam-Huber (1946-2024).

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